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FUNDAMENTALS OF ELECTRONIC MOTION

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equation. Both these equations have applications in many other types of problems besides those having to do with electric fields.[1]

Problem 1.1. Show that Coulomb's law for the force between two point charges, $F = Q_1Q_2/4\pi\epsilon_0 r^2$, may be derived from Eq. (1.2) and Gauss' law.

Problem 1.2. Find the electric force per meter between two parallel cylindrical electron beams of small radius, each having a charge $-q$ coulombs per meter, separated a distance x.

Problem 1.3. If the velocity of the electrons in Prob. 1.2 is v meters per second, the current in each beam is $-qv$ amperes. What is the magnetic field at one beam due to the presence of the other? The force in newtons per meter on a filamentary current in a magnetic field is given by the product of the current in amperes and the magnetic flux density in webers per square meter. Find the magnetic force on one of the beams due to the presence of the other.

For what electron velocity will the magnetic force exactly equal the electric force found in Prob. 1.2? Will the two forces add or tend to cancel?

Problem 1.4. A coaxial transmission line consists of two concentric cylinders with a ratio of outer to inner radius of 2.718. Find the density of energy storage in the electric field at a radius r for a potential V between the cylinders. Integrate this to find the stored energy per meter length of the line.

Find the stored energy per meter of line in the magnetic field when a constant current I flows in the inner conductor, returning through the outer conductor. Find the ratio V to I for the stored energy in the electric field to be equal to the stored energy in the magnetic field.

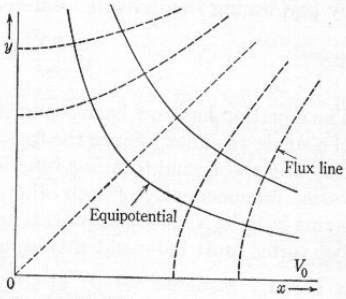

FIG. 1.10. The two-dimensional field inside a conducting right angle is obtained by solving Laplace's equation.

1.3. Solutions of Laplace's and Poisson's Equations. Although the solution of field problems is not going to be of primary concern in this book, it will be helpful to look at a few simple solutions of Laplace's and Poisson's equations.

Solutions of Laplace's Equation in Rectangular Coordinates. Let us, for example, solve Laplace's equation in the charge-free region inside a right angle formed by two conducting planes as shown in Fig. 1.10. This is a two-dimensional field; *i.e.*, there will be no variation of potential in the z direction. The boundary conditions are that the potential must be a constant value V_0 when $x = 0$ and when $y = 0$.

Sometimes Laplace's equation may be solved directly with the available information. More often, however, various solutions must be obtained

[1] The solutions to Laplace's equation have been very completely worked out in many coordinate systems. Sources of additional information on this matter are W. R. Smythe, "Static and Dynamic Electricity," McGraw-Hill, New York, 1950; and E. Weber, "Electromagnetic Fields, Theory and Applications," Wiley, New York, 1950.

Problem 1.10. Find the two-dimensional field symmetrical with respect to the x axis with the potential $V_x = e^x$ on the x axis (a) by direct substitution in Laplace's equation assuming a product type of solution, (b) by assuming a series solution $V(x,y) = a_0 + a_1y^2 + a_2y^4 + \cdots$

Problem 1.11. Show that the axially symmetric potential whose value on the z axis is $V_z = e^{kz}$ is $V(r,z) = e^{kz}J_0(kr)$ where J_0 is the zero-order Bessel function defined by the series $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

Problem 1.12. Write the general solution for Laplace's equation in cylindrical coordinates with axial symmetry and a linear variation of potential in the z direction. Fit this solution to the boundary conditions given in Fig. 1.14 for a shorted section of concentric line with a resistive inner conductor, the open end of which is held at a

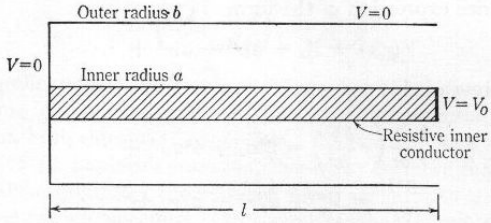

FIG. 1.14. Example of the fitting of solutions of Laplace's equation to boundary conditions (see Prob. 1.12).

potential V_0. Show that the potential between the conductors is given by

$$V = V_0 \frac{z \ln (b/r)}{l \ln (b/a)}$$

Sketch the shape of the flux lines and equipotentials.

Problem 1.13. A long cylindrical beam of electrons moving in an axial direction has a radius r_0 and a uniform charge density $-\rho_0$. Assume the potential on the outer surface of the beam is V_0. Find and plot the potential within the beam for various values of ρ_0. (In order to make these curves as general as possible it will be advantageous to plot them in terms of normalized dimensionless variables. For example, V/V_0 may be plotted against r/r_0 for various values of ρ_0/ρ_n where ρ_n is a normalizing or standard charge density. Show that one convenient value for ρ_n is $4V_0\epsilon_0/r_0^2$.)

If the electrons all leave a cathode at zero potential, how will the velocity in the center of the beam compare with that of the electrons on the outer surface of the beam?

1.4. Approximate Methods of Field Determination. It is the exceptional rather than the usual state of affairs when the geometry of a field is such that the field can be exactly expressed in the form of a simple solution of the Laplace or Poisson equation. Therefore it becomes important to have some means for obtaining approximate solutions to these equations. We shall discuss briefly flux-plotting, analogue, and net-point methods.[1]

[1] These methods are discussed in more detail in the very thorough book by E. Weber mentioned earlier.

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Flux Plotting. One of the most convenient methods for quickly determining approximate two-dimensional field configurations is that of flux plotting. Flux plotting is a means of sketching a plot of flux lines and equipotentials by successive approximations. The justification for this method stems from the flux-tube concept.

Imagine a tube of space bounded by surfaces parallel to flux lines. Since no flux can enter or leave through these surfaces, the total amount of enclosed flux is the same at all points along the tube (Fig. 1.15). The flux density, and hence the electric field strength, is inversely proportional to the cross-sectional area. Thus the spacing of equipotential surfaces with equal voltage increments between them is proportional to the area of the flux tube.

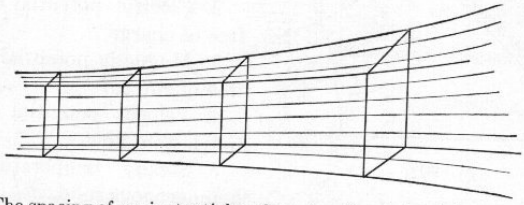

FIG. 1.15. The spacing of equipotential surfaces in a flux tube with a constant voltage increment between them is proportional to the cross-sectional area of the flux tube.

In a two-dimensional field problem where one dimension of the flux tube remains constant, the spacing between equipotentials is proportional to the spacing between flux lines. Furthermore, the two sets of surfaces are orthogonal (everywhere perpendicular) to each other. The method of flux plotting applied to a two-dimensional field consists of the freehand sketching of flux lines and equipotentials in such fashion that the space within which Laplace's equation is to be satisfied is divided into small squarelike regions. These regions are termed curvilinear squares. The ultimate test of whether or not a geometrical figure is a curvilinear square is if, on finer and finer subdivision, the subdivisions become more and more like true squares.

Very often it is possible to start off such a sketch by locating known flux lines or equipotentials, such as lines of symmetry. Several guiding principles help get the plot started. Flux lines approach conducting surfaces perpendicularly. Equipotential contours close to conducting surfaces tend to have the same shapes as these surfaces. Flux lines and equipotentials are either along or perpendicular to lines of symmetry.

Once started, the sketch must be corrected and positions of flux lines and equipotentials adjusted until the entire region is filled with curvilinear squares. A flux plot sketched in this fashion is shown in Fig. 1.16.

This method can be adapted to certain three-dimensional field problems, such as axially symmetric fields, and to certain fields which do not satisfy Laplace's equation. In such cases the subdivisions are not curvilinear squares but possess other properties.

Analogue Methods. Laplace's equation is found to be the mathematical relation which describes a number of physical situations. In some common examples the dependent variable may correspond to

1. Gravitational potential in space free of matter
2. Electric potential in space free of charge
3. Magnetic potential in space free of current
4. Velocity potential of irrotational fluid flow
5. Steady temperature in a homogeneous solid
6. Electric potential in a homogeneous conducting medium
7. Height of an elastic membrane with small displacements from a plane (two-dimensional form only)

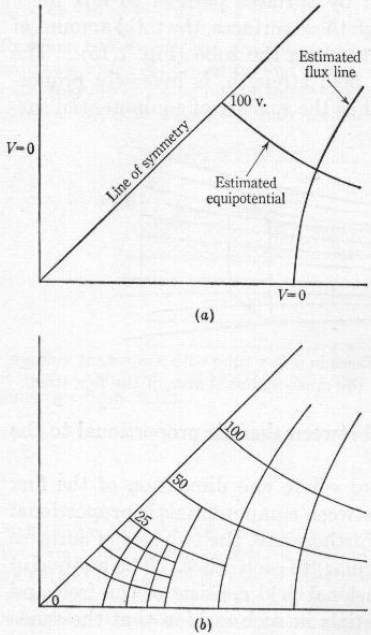

FIG. 1.16. Flux-plotting method of field determination. (*a*) First stages of flux plot of the region within a conducting right angle; (*b*) the flux plot in a more advanced stage.

The method of analogues in engineering is simply that of obtaining experimentally a solution to a problem which is described by the same mathematical equations as the desired problem but which, for some reason, is more easily set up or dealt with. Current-flow, fluid-flow, and elastic-membrane analogues have been found particularly useful in obtaining solutions to static electric and magnetic field problems. They are most conveniently applied to two-dimensional fields or axially symmetric fields.

Current-flow models are conveniently made using imperfectly conducting solids or electrolytes as the homogeneous medium. For a two-dimensional field problem a thin sheet of a solid medium or a shallow level of electrolyte in a flat tank is suitable. The model may often be set up for only a portion of the region to be examined, by making use of surfaces of symmetry. For example, Fig. 1.17 indicates how a model might be

set up to determine the field configuration in a triode. By making full use of symmetry properties, a model for one small segment of the interelectrode space suffices to determine the entire field. Conducting surfaces of electrodes are represented by conducting boundaries in the model. In this case equipotentials must be perpendicular to surfaces of symmetry.

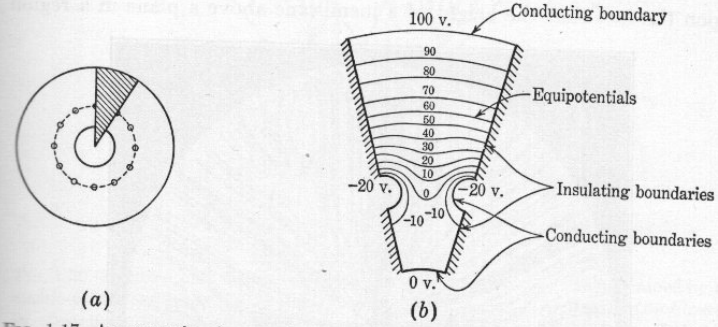

FIG. 1.17. An example of an analogue method of field determination. (*a*) Section of triode represented by model; (*b*) arrangement of boundaries in a current-flow model.

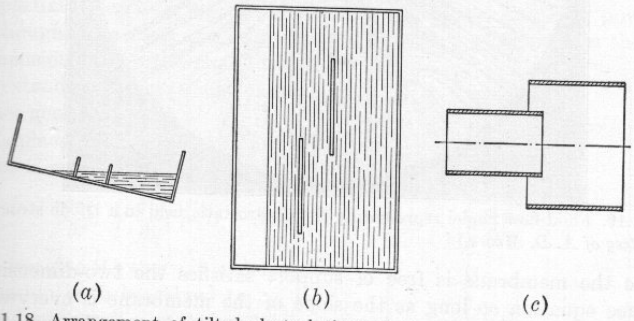

FIG. 1.18. Arrangement of tilted electrolytic tank to represent axially symmetric field of a simple electron lens. (*a*) Side view; (*b*) top view; (*c*) cylindrical electrodes represented.

This is insured by making the symmetry lines insulating boundaries. With the electrodes held at appropriate potentials the shape of the equipotentials, which may be traced with a probe, is the same in the analogue as in the original electrostatic problem.

This technique may be simply extended to the case of axially symmetric fields by using a wedge-shaped section of imperfectly conducting material or a tilted electrolytic tank as indicated in Fig. 1.18.

Two-dimensional fluid-flow models are particularly effective from the standpoint of being able to visualize the fields. The technique used here is to represent, say, positive charges by fluid outlets. Flow lines (flux lines) may be observed by placing small crystals of soluble dye throughout the region (Fig. 1.19).[1]

The elastic-membrane analogue for two-dimensional fields depends upon the fact that the height of a membrane above a plane in a region

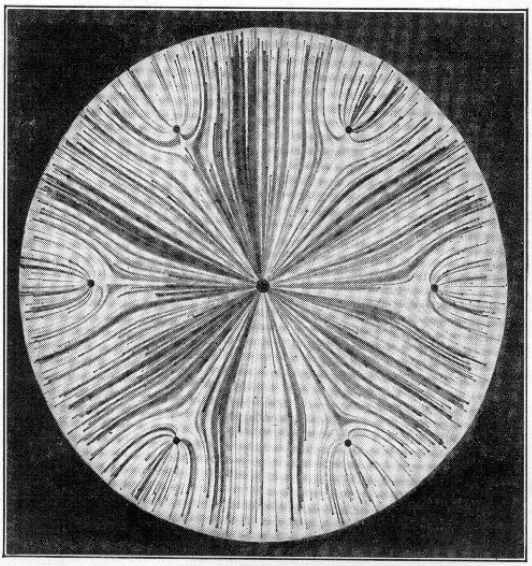

FIG. 1.19. Fluid-flow model representing the electrostatic field in a triode structure. (*Courtesy of A. D. Moore.*)

where the membrane is free of support satisfies the two-dimensional Laplace equation so long as the slope of the membrane is everywhere small. In the membrane model conducting electrodes at constant potential are represented by supports of uniform height, the height being proportional to the negative of the potential. Lines of symmetry which are parallel to flux lines are represented by free edges of the membrane. For the purpose of field determination this method is probably less satisfactory than the others discussed, but it has the advantage that small spheres placed on the membrane at the height corresponding to zero potential will follow paths similar to those of electrons in the electric field analogue.

[1] A. D. Moore, Fields from Fluid Flow Mappers, *J. Appl. Phys.*, Vol. 20, pp. 790–804, August, 1949.

This method of obtaining electron paths will be mentioned in the discussion of electron motion in static electric fields (Fig. 1.20).

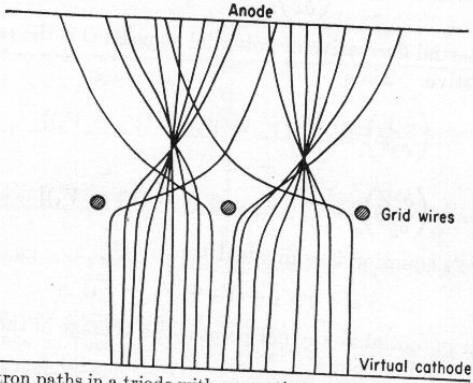

FIG. 1.20. Electron paths in a triode with a negative grid voltage, as determined by an elastic-membrane model. (*Adapted from P. H. J. A. Kleynen, The Motion of an Electron in Two-dimensional Electrostatic Fields, Philips Tech. Rev., Vol. 2, pp.* 338–345, *November,* 1937.)

Net-point Methods. Net-point or relaxation methods for the solution of partial differential equations are very convenient and very powerful. Although the examples given here are for Laplace's equation in the two-dimensional form the method may be extended to other equations and other geometries.

Laplace's equation in two dimensions is written

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (1.43)$$

We imagine the space in which this equation is to be solved to be divided into small squares by a lattice or net and approximate the derivatives of the potential in terms of the potentials at the points of the lattice. For example, following the labeling of points in Fig. 1.21, the first partial derivative of potential with respect to x at point a is approximated by

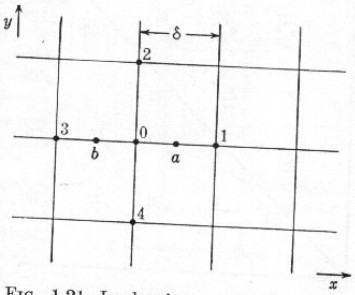

FIG. 1.21. Laplace's equation in difference form is equivalent to the statement that the potential at point O is the average of the potentials at the four surrounding points.

$$\left(\frac{\partial V}{\partial x}\right)_a = \frac{V_1 - V_0}{\delta} \qquad (1.44)$$

Similarly, at point b,

$$\left(\frac{\partial V}{\partial x}\right)_b = \frac{V_0 - V_3}{\delta} \tag{1.45}$$

The second partial derivative of potential at point O is the rate of change of first derivative. Thus

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_o = \frac{1}{\delta^2}\left[(V_1 - V_0) - (V_0 - V_3)\right] \tag{1.46}$$

Similarly

$$\left(\frac{\partial^2 V}{\partial y^2}\right)_o = \frac{1}{\delta^2}\left[(V_2 - V_0) - (V_0 - V_4)\right] \tag{1.47}$$

Thus Laplace's equation is equivalent to

$$V_0 = \tfrac{1}{4}(V_1 + V_2 + V_3 + V_4) \tag{1.48}$$

In words, the potential at any net point is the average of the potentials at the four surrounding points.

This may be used as the basis for a method of field determination. Consider as an example the field configuration within the right angle formed by two conducting planes as shown in Fig. 1.22. Let the poten-

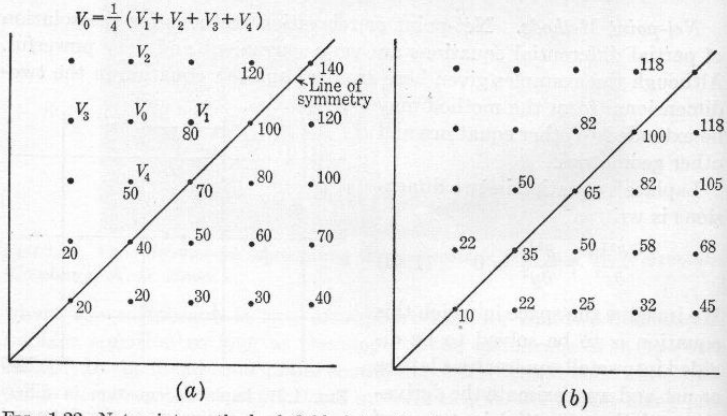

FIG. 1.22. Net-point method of field determination in a conducting right angle. (a) Preliminary assignment of potentials to net points; (b) potentials after first correction.

tial of the two conducting planes be taken to be zero and the potential of one point on the line of symmetry arbitrarily held at 100 volts. A preliminary assignment of potentials to the other net points is made as shown, using intuition and experience as guides. Because of the symmetry condition only half the region need be treated.

Now, starting at one corner and progressing through the region, the potential estimate at each net point is revised in accordance with Eq.

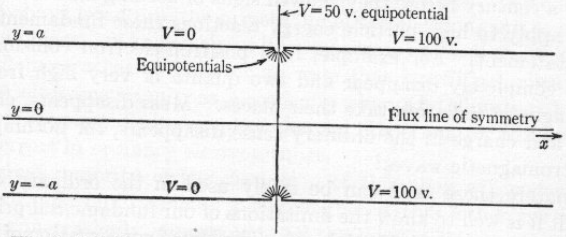

FIG. 1.23. The field in the region defined by these four planes is to be found by flux plotting.

(1.48). These new values are not used to correct the preliminary potential assignments as the calculation progresses, but rather form a new potential net which is then revised in the same manner. When the values no longer change appreciably in successive trials, the net potentials may be considered correct within an accuracy determined by the mesh size. Equipotentials may then be drawn in and flux lines constructed orthogonal to these to complete the field plot.

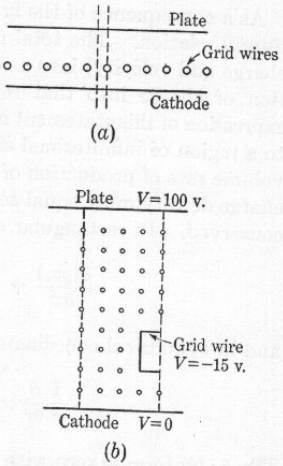

FIG. 1.24. Illustration of the net-point method of field determination. (*a*) Section of a planar triode to be examined by net-point method; (*b*) enlarged section showing net points.

Problem 1.14. Complete the two-dimensional flux plot of the region enclosed by the four plane electrodes shown in Fig. 1.23. From the flux plot obtain a curve of potential along the x axis as a function of x. At what values of x does the tangent to this curve at $x = 0$ intersect the lines $V = 0$ and $V = 100$?

This particular field happens to be one which can be rather easily solved analytically. As a check on the accuracy of your flux plotting, compare the two values of x called for above with those obtained by analysis which are a and $-a$.

Problem 1.15. Figure 1.24 represents a section of a triode in which the potential is to be found by the net-point method. Carry through successive corrections until the potentials by the net points do not change by more than a volt in two successive recalculations. Sketch the equipotentials.

What is the electric field at the cathode surface directly behind a grid wire? What is the electric field at the cathode surface midway between grid wires? Will any electrons leave the cathode surface? Is 15 volts above or below the cutoff potential for this triode?