Distance between a point and an isoline of a 2D function

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Credit

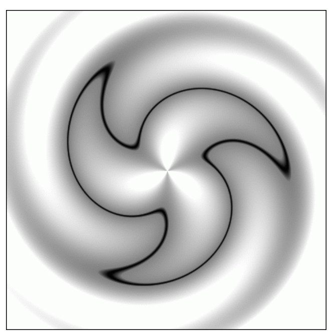
See Iñigo Quilez' page at http://www.iquilezles.org/www/articles/distance/distance.htm.

The problem

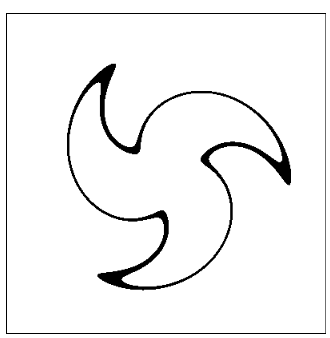
Suppose we have a two-dimensional scalar function $f(x, y)$ and want to draw the isolines (lines of constant f), specifically the $f = 0$ isoline. As an example, we will consider the polar function

$$f(r, \theta) = r - 1 + \frac{1}{2}\sin\left(3\theta + 2r^2\right)$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$, as usual. Below is a gradient plot of $|f|$, with black=small $|f|$ and white=large $|f|$, over the range $x = -2 \ldots 2, y = -2 \ldots 2$. The $f = 0$ isoline corresponds to the black "triple swirl" shape. $f < 0$ inside the shape and $f > 0$ outside.



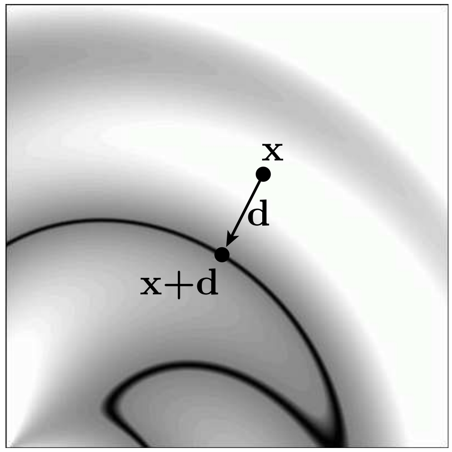
To draw only the $f = 0$ isoline, we might colour a pixel black only if $|f|$ is less than a certain limit. For example, if the limit is 0.05, we obtain the following shape.



This follows the isoline closely, but results in a shape with varying width - suppose we wanted a shape with constant width instead. We need an estimate of the *closest* distance from a particular point (pixel) to the isoline - if the black/white limit is applied to such a distance estimate, we will obtain a line of reasonably constant width.

Derivation

We will use vectors to represent points throughout. Let our chosen point be $\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, and let the point on the $f = 0$ isoline which is closest be $\mathbf{x} + \mathbf{d}$, where $\mathbf{d} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}$ is the distance vector from the point to the isoline. These are shown on the diagram below.



Let us define $f(\mathbf{x} + \mathbf{d})$ in terms of $f(\mathbf{x})$ using the 2D Taylor series. "*HOT*" represents higher-order terms which will be ignored.

$$\begin{aligned} f(\mathbf{x}+\mathbf{d}) &= f(x+\delta x, y+\delta y) = f(x,y) + \left(\frac{\partial f(x,y)}{\partial x}\delta x + \frac{\partial f(x,y)}{\partial y}\delta y\right) + HOT \\ &\cong f(\mathbf{x}) + \left(\frac{\partial f(x,y)}{\partial x}\hat{\mathbf{i}} + \frac{\partial f(x,y)}{\partial y}\hat{\mathbf{j}}\right)\cdot \mathbf{d} \\ &= f(\mathbf{x}) + \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}}\right) f(\mathbf{x}) \cdot \mathbf{d} \\ &= f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{d} \end{aligned} \tag{1}$$

The point $\mathbf{x} + \mathbf{d}$ lies on the isoline, therefore $|f(\mathbf{x}+\mathbf{d})| = 0$ (note the modulus). From (1), we have

$$0 = |f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{d}| \tag{2}$$

We now need to use the *triangle inequality*. Simply put, this states that the modulus of the sum of two numbers must always be less than the sum of their individual moduli:

$$|a+b| \leq |a| + |b| \tag{3}$$

For the purposes of this derivation, it is convenient to use a slightly different form:

$$|a+b| \geq |a| - |b| \tag{4}$$

(Later, we will use the original triangle inequality (3) and show it gives the same result.) Applying our modified triangle inequality (4) to (2), we obtain

$$\begin{aligned} |f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{d}| &\geq |f(\mathbf{x})| - |\nabla f(\mathbf{x}) \cdot \mathbf{d}| \\ 0 &\geq |f(\mathbf{x})| - |\nabla f(\mathbf{x}) \cdot \mathbf{d}| \end{aligned} \tag{5}$$

Next, we need to use an inequality derived from the vector dot product. The dot product of two vectors $\mathbf{a}$ & $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \tag{6}$$

$\cos\theta$ has values from $-1\ldots+1$. Therefore, $\mathbf{a}\cdot\mathbf{b}$ can take values ranging from $-|\mathbf{a}||\mathbf{b}|$ to $|\mathbf{a}||\mathbf{b}|$. Taking the modulus, $|\mathbf{a}\cdot\mathbf{b}|$ has values from 0 to $|\mathbf{a}||\mathbf{b}|$. Therefore,

$$|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}| \tag{7}$$

Using this inequality on $|\nabla f(\mathbf{x})\cdot\mathbf{d}|$, we have

$$\begin{aligned} |\nabla f(\mathbf{x})\cdot\mathbf{d}| &\leq |\nabla f(\mathbf{x})||\mathbf{d}| \\ -|\nabla f(\mathbf{x})\cdot\mathbf{d}| &\geq -|\nabla f(\mathbf{x})||\mathbf{d}| \\ |f(\mathbf{x})| - |\nabla f(\mathbf{x})\cdot\mathbf{d}| &\geq |f(\mathbf{x})| - |\nabla f(\mathbf{x})||\mathbf{d}| \end{aligned} \tag{8}$$

We know from (5) that $0 \geq |f(\mathbf{x})| - |\nabla f(\mathbf{x})\cdot\mathbf{d}|$, therefore

$$0 \geq |f(\mathbf{x})| - |\nabla f(\mathbf{x})||\mathbf{d}| \tag{9}$$

We can rearrange this to obtain

$$\begin{aligned} |\nabla f(\mathbf{x})||\mathbf{d}| &\geq |f(\mathbf{x})| \\ |\mathbf{d}| &\geq \frac{|f(\mathbf{x})|}{|\nabla f(\mathbf{x})|} \end{aligned} \tag{10}$$

This is an estimate for the minimum distance between the point $\mathbf{x}$ and the $f = 0$ isoline of the function $f(\mathbf{x})$.

If we wanted the distance to a different isoline, in general $f = v$, simply replace $|f(\mathbf{x})|$ by $|f(\mathbf{x}) - v|$ in (10).

Aside: if we had used the original triangle inequality (3), we would obtain

$$|\mathbf{d}| \geq -\frac{|f(\mathbf{x})|}{|\nabla f(\mathbf{x})|} \tag{11}$$

which is essentially the same result except for the minus sign, which can probably be ignored because we are only interested in the moduli of the various terms. However, using the modified triangle inequality (4) results in a simpler calculation.

Example

Let us use (10) to draw isolines for the "triple swirl" function. To calculate $\nabla f(\mathbf{x})$, we will use a central differences approximation since it avoids having to calculate the partial derivatives analytically.

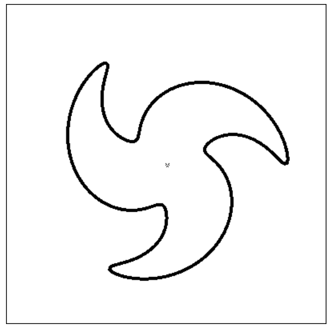
$$\begin{aligned} \nabla f(\mathbf{x}) &= \nabla f(x,y) \\ &= \frac{\partial f(x,y)}{\partial x}\hat{\mathbf{i}} + \frac{\partial f(x,y)}{\partial y}\hat{\mathbf{j}} \\ &= \frac{f(x+\delta x,y) - f(x-\delta x,y)}{2\delta x}\hat{\mathbf{i}} + \frac{f(x,y+\delta y) - f(x,y-\delta y)}{2\delta y}\hat{\mathbf{j}} \end{aligned} \tag{12}$$

Note that the δx & δy in (12) are not the same as the components of the distance vector $\mathbf{d}$ - they are simply small increments in the x and y directions. If we let $\delta x = \delta y = \delta$, (12) becomes

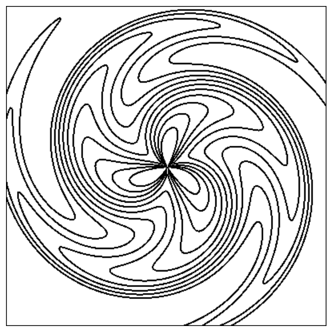
$$\nabla f(\mathbf{x}) = \frac{1}{2\delta}\left\{[f(x+\delta,y) - f(x-\delta,y)]\,\hat{\mathbf{i}} + [f(x,y+\delta) - f(x,y-\delta)]\,\hat{\mathbf{j}}\right\} \tag{13}$$

We choose δ to be small compared with any features of interest - in this example, $\delta = 0.01$.

If we now colour every pixel black when the distance calculated using (10) is less than a limit (in this case, 0.02), we obtain the following representation of the $f = 0$ isoline.



This is a much better representation compared to the “simple” scheme used back at the beginning. The line is a constant width and follows the isoline perfectly. If we plot several isolines (from -1 to 1 step 0.2), we obtain:



Again, all lines are a constant width, regardless of isoline or position. Instead of a black & white representation, we can map either the distance or the isoline value to a colour gradient and obtain some interesting results: