The University of Tennessee at Martin

MARTIN School of Engineering

Stresses in Rotating Disks

Lecture 16

Engineering 473

Machine Design

Summary of Axisymmetric Equations

Equilibrium Equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

Strain-Displacement Equations

$$\varepsilon_r = \frac{du}{dr}, \ \varepsilon_\theta = \frac{u}{r}$$

Constitutive Equations

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \upsilon\sigma_\theta\right)$$

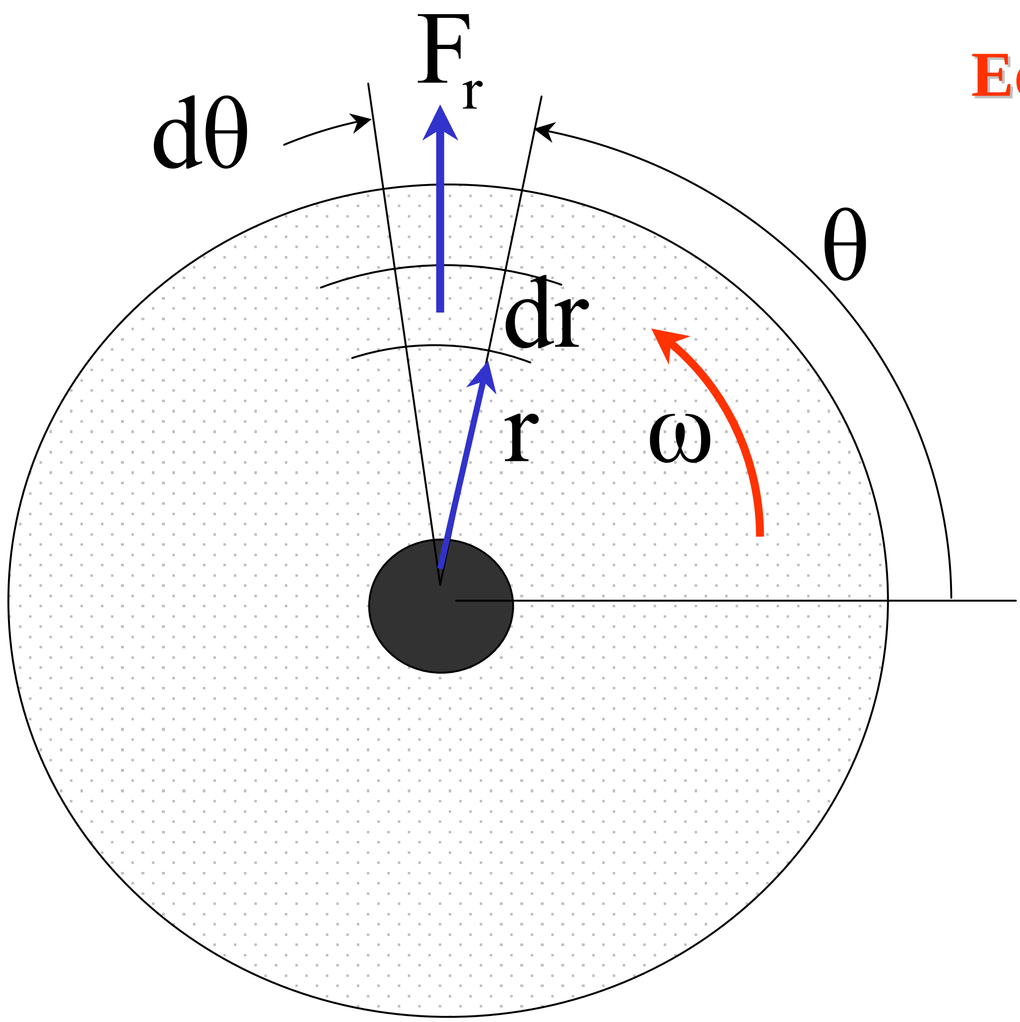
$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \upsilon\sigma_r\right)$$

or

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \upsilon\varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \upsilon\varepsilon_r\right)$$

Rotating Disk



Equilibrium Diff Equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

↓

$$F_r = \rho r \omega^2$$

↓

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho r \omega^2 = 0$$

$F_r \equiv$ radial body force per unit volume

Displacement Base Equilibrium Equation

Equilibrium Equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho r \omega^2 = 0$$

Constitutive Equations

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \upsilon \varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \upsilon \varepsilon_r\right)$$

Combining the equilibrium and constitutive equations yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^r} = -\left(1-\upsilon^2\right)\frac{\rho r \omega^2}{E}$$

This equation is the differential equation of equilibrium written in terms of the radial displacement component.

General Solution

Differential Equation of Equilibrium

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^r} = -\left(1 - \upsilon^2\right)\frac{\rho r \omega^2}{E}$$

Homogeneous Solution

$$u_h = C_1 r + \frac{C_2}{r}$$

The homogeneous solution is the same as the general solution for the thick walled cylinder.

Particular Solution

$$u_p = -\left(1 - \upsilon^2\right)\frac{\rho r^3 \omega^2}{8E}$$

General Solution

$$u = C_1 r + \frac{C_2}{r} - \left(1 - \upsilon^2\right)\frac{\rho r^3 \omega^2}{8E}$$

Stress Distributions

Constitutive Equations

$$\sigma_r = \frac{E}{1-\upsilon^2}\left(\varepsilon_r + \upsilon\varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\upsilon^2}\left(\varepsilon_\theta + \upsilon\varepsilon_r\right)$$

General Solution - Displacement

$$u = C_1 r + \frac{C_2}{r} - \left(1-\upsilon^2\right)\frac{\rho r^3\omega^2}{8E}$$

Displacement Based

$$\sigma_r = \frac{E}{1-\upsilon^2}\left(\frac{du}{dr} + \upsilon\frac{u}{r}\right)$$

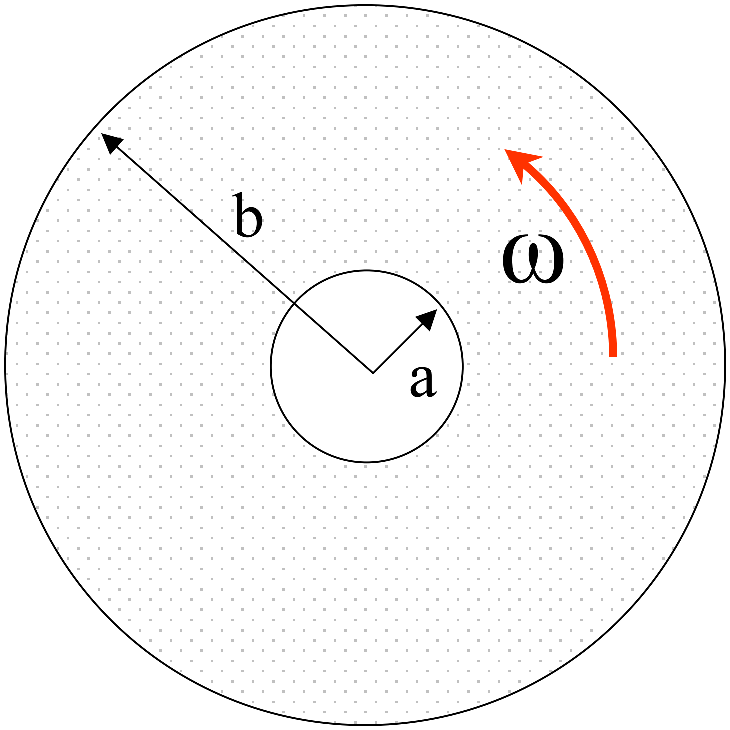
$$\sigma_\theta = \frac{E}{1-\upsilon^2}\left(\frac{u}{r} + \upsilon\frac{du}{dr}\right)$$

General Solution - Stress

$$\sigma_r = \frac{E}{1-\upsilon^2}\left[\frac{-(3+\upsilon)\left(1-\upsilon^2\right)\rho r^2\omega^2}{8E} + (1+\upsilon)C_1 - (1-\upsilon)\frac{C_2}{r^2}\right]$$

$$\sigma_\theta = \frac{E}{1-\upsilon^2}\left[\frac{-(1+3\upsilon)\left(1-\upsilon^2\right)\rho r^2\omega^2}{8E} + (1+\upsilon)C_1 + (1-\upsilon)\frac{C_2}{r^2}\right]$$

Annular Rotating Disk



Boundary Conditions

$$\sigma_r(a) = 0$$

$$\sigma_r(b) = 0$$

This disk has a hole in the center.

Constant Determination for Annular Rotating Disk

$$\sigma_r(a) = \frac{E}{1-\upsilon^2}\left[\frac{-(3+\upsilon)(1-\upsilon^2)\rho a^2\omega^2}{8E} + (1+\upsilon)C_1 - (1-\upsilon)\frac{C_2}{a^2}\right]$$
$$= 0$$

$$\sigma_r(b) = \frac{E}{1-\upsilon^2}\left[\frac{-(3+\upsilon)(1-\upsilon^2)\rho b^2\omega^2}{8E} + (1+\upsilon)C_1 - (1-\upsilon)\frac{C_2}{b^2}\right]$$
$$= 0$$

Multiplying the top equation by a^2 and the bottom by b^2 and then subtracting the two equations yields

$$C_1 = \rho\omega^2\frac{(a^2+b^2)}{E}\frac{(1-\upsilon)(3+\upsilon)}{8}$$

Constant Determination
(Continued)

$$\sigma_r(a) = \frac{E}{1-\upsilon^2}\left[\frac{-(3+\upsilon)(1-\upsilon^2)\rho a^2\omega^2}{8E} + (1+\upsilon)C_1 - (1-\upsilon)\frac{C_2}{a^2}\right] = 0$$

$$\sigma_r(b) = \frac{E}{1-\upsilon^2}\left[\frac{-(3+\upsilon)(1-\upsilon^2)\rho b^2\omega^2}{8E} + (1+\upsilon)C_1 - (1-\upsilon)\frac{C_2}{b^2}\right] = 0$$

$$C_1 = \rho\omega^2\frac{(a^2+b^2)}{E}\frac{(1-\upsilon)(3+\upsilon)}{8}$$

$$C_2 = \rho\omega^2\left(\frac{a^2b^2}{E}\right)\frac{(1+\upsilon)(3+\upsilon)}{8}$$

Annular Rotating Disk Equations

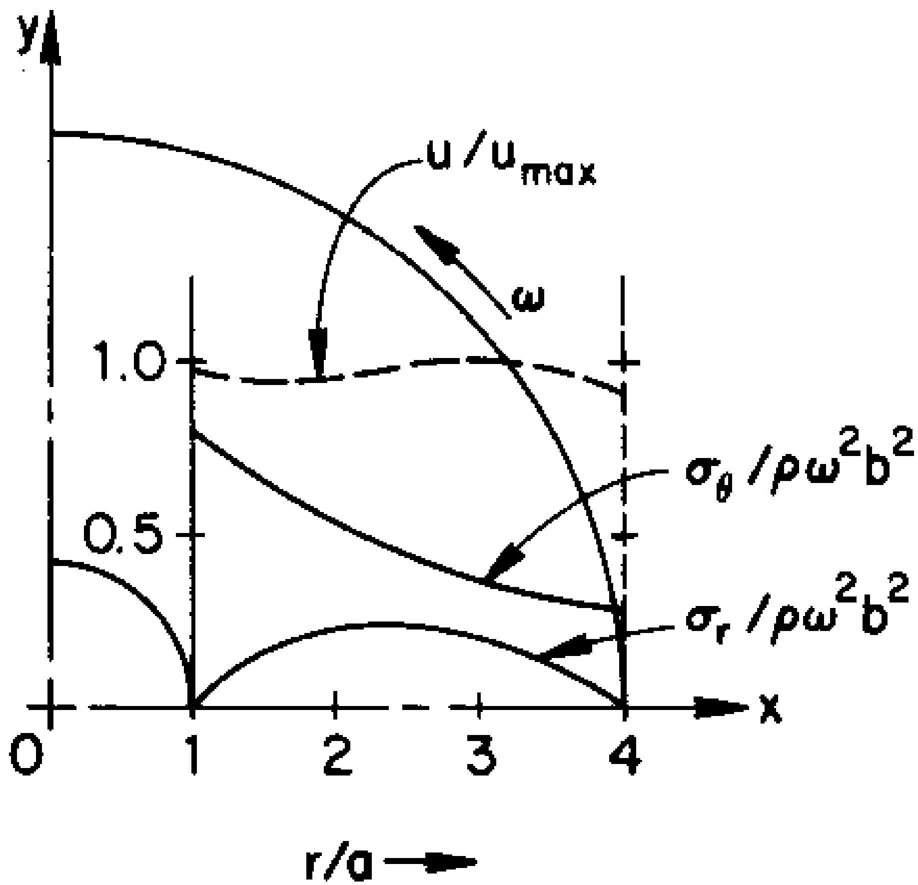
$$\sigma_r = \frac{3+\upsilon}{8}\left(a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2}\right)\rho\omega^2$$

$$\sigma_\theta = \frac{3+\upsilon}{8}\left(a^2 + b^2 - \frac{1+3\upsilon}{3+\upsilon} r^2 + \frac{a^2 b^2}{r^2}\right)\rho\omega^2$$

$$u = \frac{(3+\upsilon)(1-\upsilon)}{8E}\left(a^2 + b^2 - \frac{1+\upsilon}{3+\upsilon} r^2 + \frac{1+\upsilon}{1-\upsilon}\frac{a^2 b^2}{r^2}\right)\rho r\omega^2$$

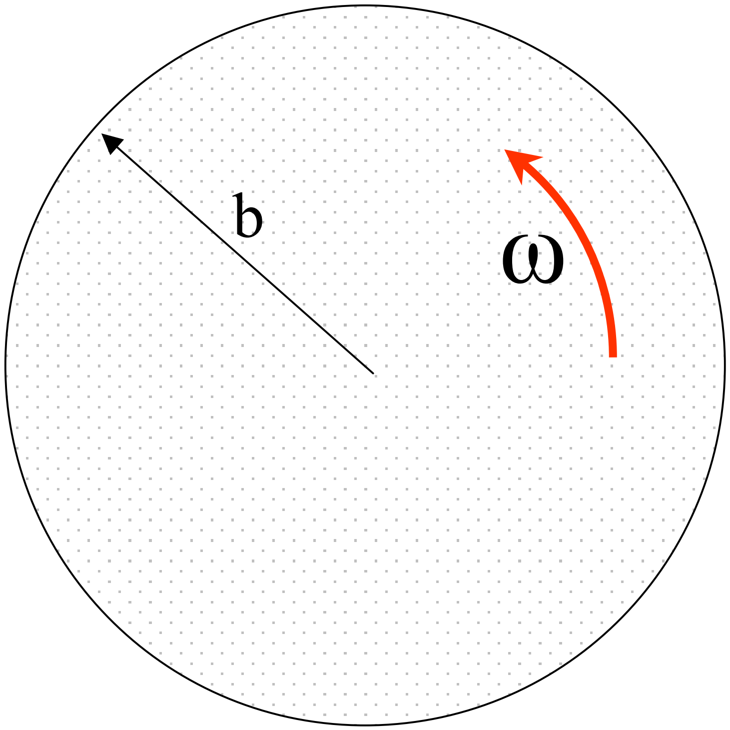
Note that r=a and r=b, that the radial stress component is zero.

Stress and Displacement Variation Through the Thickness



Ugural, Fig. 8.6

Solid Rotating Disk



Boundary Conditions

$$\sigma(b) = 0$$

$$u(0) = 0$$

Solid Rotating Disk
(Continued)

$$u = C_1 r + \frac{C_2}{r} - \left(1-\nu^2\right)\frac{\rho r^3 \omega^2}{8E}$$

$$\sigma_r = \frac{E}{1-\upsilon^2}\left[\frac{-(3+\upsilon)\left(1-\upsilon^2\right)\rho r^2 \omega^2}{8E} + (1+\upsilon)C_1 - (1-\upsilon)\frac{C_2}{r^2}\right]$$

Since the displacement must be finite at r = 0, $C_2 = 0$

$$C_1 = \rho\omega^2 \frac{b^2}{E}\frac{(1-\upsilon)(3+\upsilon)}{8}$$

Solid Rotating Disk Stress and Displacement Equations

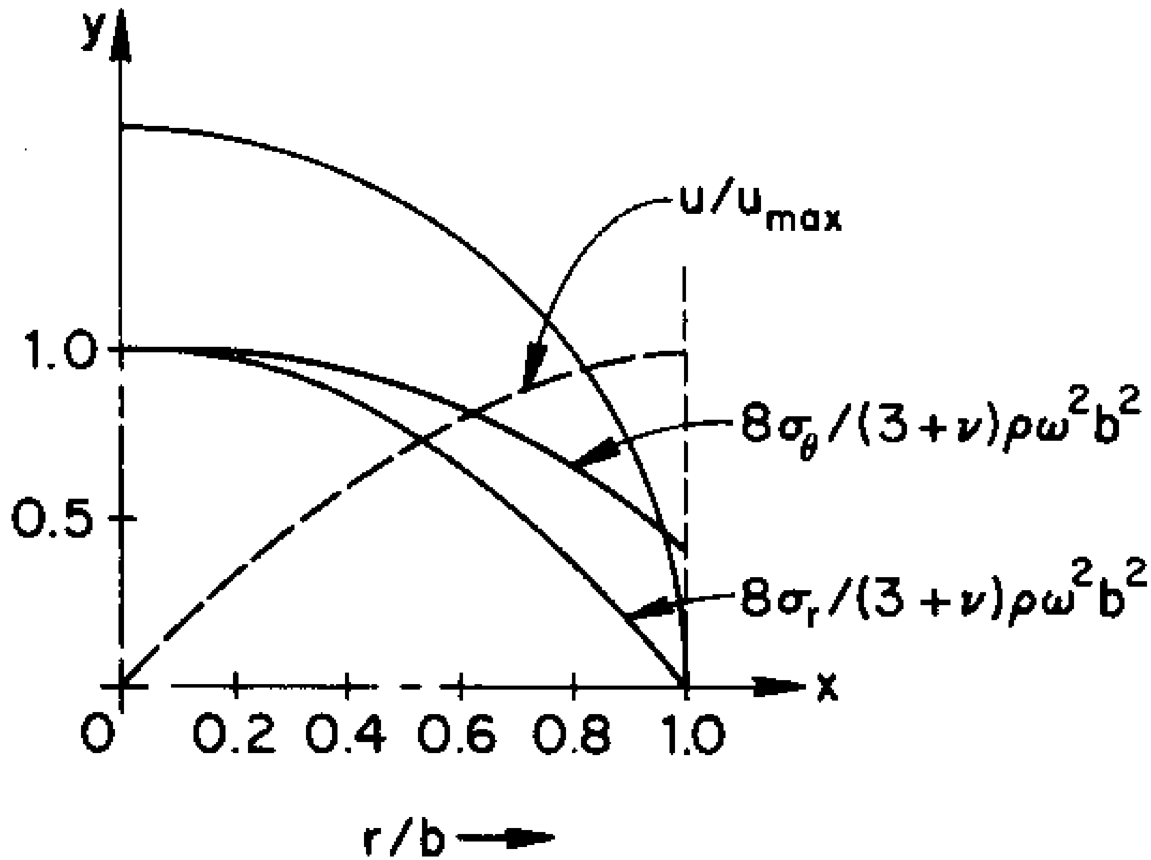
$$\sigma_r = \frac{3+\upsilon}{8}\left(b^2 - r^2\right)\rho\omega^2$$

$$\sigma_\theta = \frac{3+\upsilon}{8}\left(b^2 - \frac{1+3\upsilon}{3+\upsilon}r^2\right)\rho\omega^2$$

$$u = \frac{1-\upsilon}{8E}\left[(3+\upsilon)b^2 - (1+\upsilon)r^2\right]\rho r\omega^2$$

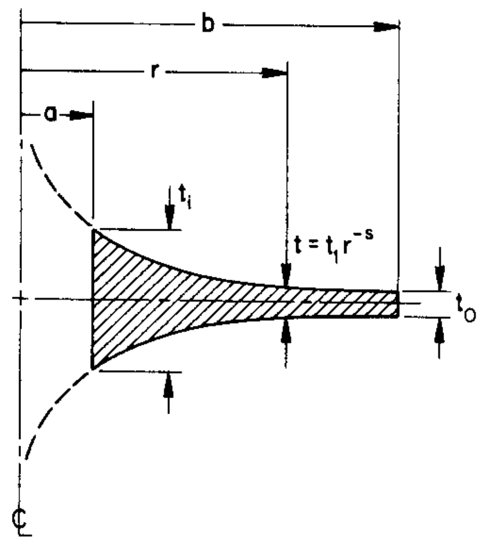
Note that these equations satisfy the boundary conditions.

Stress and Displacement Variation

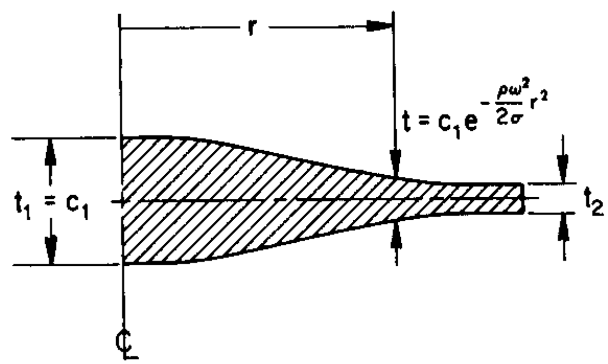


Other Solutions

Solutions to the governing differential equations exist for variable thickness geometries and for constant stress conditions.



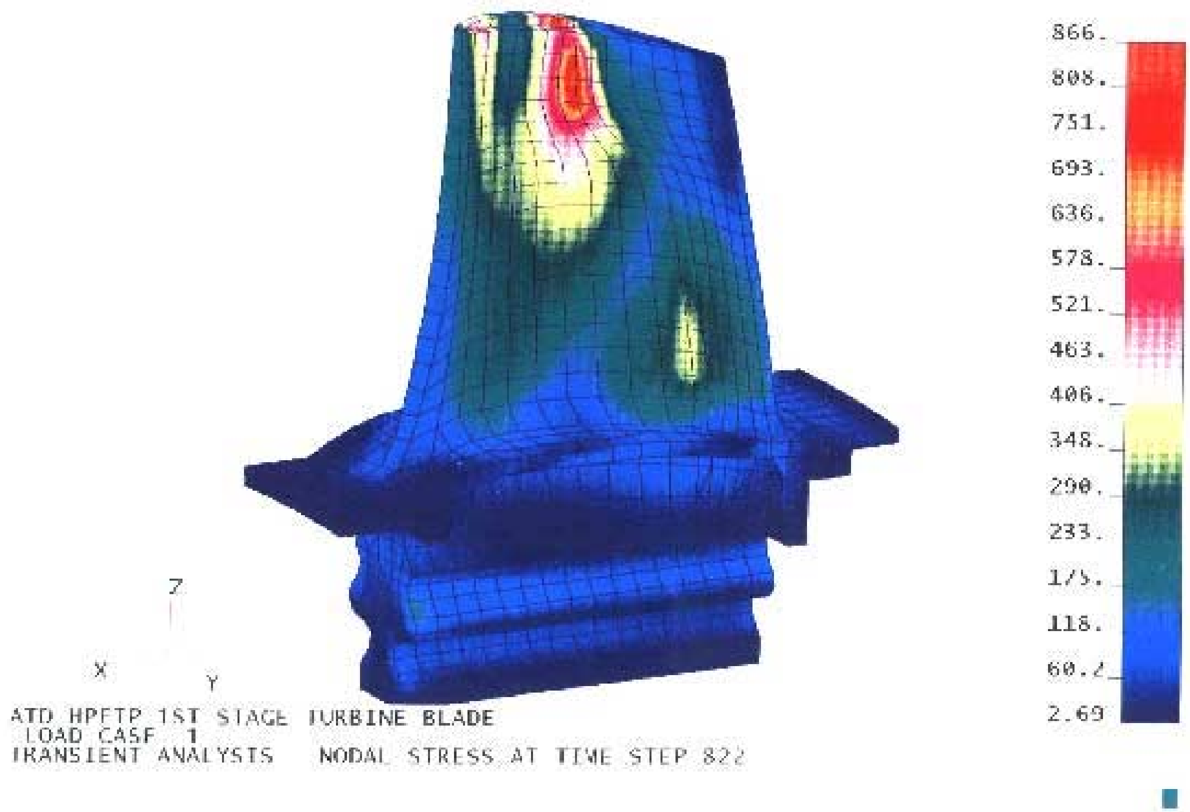
Variable Thickness



Constant Stress

Ugural, Fig. 8.8 & 8.9

Complex Geometries



Complex geometries must be solved using numerical methods.

Assignment

A flat 20 inch outer diameter, 4 inch inner diameter, and 3 inch thick steel disk is shrunk onto a steel shaft. If the assembly is to run safely at 6900 rpm, determine: (a) the required interference (inches), (b) the maximum stress when not rotating, and (c) the maximum stress when rotating. The material properties are ρ=0.00072 lb-sec^2/in^4, E=30x10^6 psi, and ν=0.3.